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GENERALIZED CONJUGATE MATRICES.

By Philip Franklin.

The notion of conjugate matrices, which originated with O. Taber* and was applied in a recent paper by A. A. Bennett,† may be described in the following terms. If, corresponding to a given matrix M_1 of order n there exist $n - 1$ matrices of the same order satisfying the conditions:

1. They have the same characteristic equation as M_1;
2. They are commutative with respect to multiplication;
3. The symmetric functions ΣM_i, $\Sigma M_i M_j$, $\cdots$, $M_1 \cdot M_2 \cdot \cdots M_n$ formed from the matrix M_1 and these $n - 1$ matrices are scalars and equal to the corresponding functions of the n scalar roots of the characteristic equation of M_1;

these $n - 1$ matrices are called the $n - 1$ *conjugates* of M_1.

In the case where the roots of the characteristic equation of the given matrix are all distinct, the existence of the conjugate matrices is demonstrated‡ by noting that if r_1, r_2, $\cdots$ r_n are these n distinct roots, the matrix

$$R_1 = \begin{Vmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & r_n \end{Vmatrix} \tag{1}$$

has the same elementary divisors as the given matrix M_1. Consequently§ a non-singular matrix P can be found such that:

$$M_1 = PR_1P^{-1}, \tag{2}$$

and the $n - 1$ matrices

$$M_i = PR_iP^{-1}, \tag{3}$$

where R_i is given by:

$$R_i = \begin{Vmatrix} r_i & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_{i+1} & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & r_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & r_1 & \cdots & 0 \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & r_{i-1} \end{Vmatrix}, \tag{4}$$

evidently satisfy the three conditions stated above.

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* Taber, O., On certain identities in the theory of matrices. Amer. Journ. Math., vol. 13 (1891), p. 159.

† Bennett, A. A., Some algebraic analogies in matric theory, these Annals, vol. 23, p. 91.

‡ Cf. Taber, l. c.

§ Bôcher, M., Introduction to Higher Algebra, p. 283.

The same method applies to a matrix whose characteristic equation has equal roots, provided it has the same elementary divisors as a matrix of the form R_1, with some of the r's equal. In the case of matrices whose characteristic equations have equal roots, but whose elementary divisors are not of this type, the method fails. In fact in this case there may fail to exist a set of $n - 1$ matrices conjugate to the given matrix.*

The purpose of this note is to define a set of *generalized conjugate* matrices, which are subject to less stringent conditions than the ordinary conjugate matrices, but which have the advantage of existing in all cases. Furthermore, they are sufficient for many of the proofs given by Taber and Bennett which use ordinary conjugate matrices.

Our *generalized conjugate* matrices differ from the ordinary conjugate matrices in that they are required to satisfy conditions (2) and (3) only. The sacrifice of condition (1) is not as violent as it at first appears, since by the use of the remaining two conditions the characteristic equation:

$$M_1 - \lambda I = 0 \tag{5}$$

reduces to:

$$(M_1 - \lambda)(M_2 - \lambda) \cdots (M_n - \lambda) = 0 \tag{6}$$

and consequently the generalized conjugate matrices are *roots* of the characteristic equation.

To set up these matrices explicitly, we shall first notice† that any matrix is equivalent to a matrix of the form:

$$S_1 = \left\| \begin{array}{cccc} S_1^1 & & & \\ & S_1^2 & & \\ & & \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix} & \\ & & & S_1^p \end{array} \right\|, \tag{7}$$

where the missing elements are zeros and the S_1^i's are blocks of terms given by:

$$S_1^i = \left\| \begin{array}{ccccc} r_i & 1 & 0 & \cdots & 0 \\ 0 & r_i & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & r_i \end{array} \right\|. \tag{8}$$

Simple roots correspond to blocks of a single term. We therefore have, analogously to (2):

$$M_1 = PS_1P^{-1}. \tag{9}$$

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* Bennett, l. c.

† Bôcher, l. c., p. 289.

In the case where S_1 consists of a single block,

$$S_1 = \begin{Vmatrix} r_1 & 1 & 0 & \cdots & 0 \\ 0 & r_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & r_1 \end{Vmatrix}, \tag{10}$$

the $n - 1$ matrices:

$$M_i = PS_iP^{-1}, \tag{11}$$

where S_i is given by

$$S_i = \begin{Vmatrix} r_1 & \omega^{i-1} & 0 & \cdots & 0 \\ 0 & r_1 & \omega^{i-1} & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & r_1 \end{Vmatrix}, \tag{12}$$

ω being a primitive nth root of unity, are evidently a set of $n - 1$ ordinary conjugates of M_1. In the general case, where M_1 satisfies the relations (9) and (7), we form the $n - 1$ matrices:

$$S_i = \begin{Vmatrix} S_i^1 & & & \\ & S_i^2 & & \\ & & \ddots & \\ & & & S_i^p \end{Vmatrix}, \tag{13}$$

where for each block S_i^j of k rows, the $k - 1$ conjugates of S_1^j, the corresponding block of S_1, appear in $k - 1$ of the matrices, and in the remaining $n - k$ blocks its place is filled by a scalar block of the form:

$$\begin{Vmatrix} r_i^{(j)} & 0 & \cdots & 0 \\ 0 & r_i^{(j)} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & r_i^{(j)} \end{Vmatrix}, \tag{14}$$

the r_i's being roots of the characteristic equation of M_1 and so selected that the $n - k$ values of r_i for the $n - k$ different values of i together with the k roots corresponding to S_1^j form the complete set of n roots of the characteristic equation of M_1. The generalized conjugate matrices of M_1 are then the $n - 1$ matrices obtained by combining (13) and (11). This is readily verified if we first notice that any rational integral function of the S_i's, $f(S_1, S_2, \cdots S_n)$, is given by:

$$f(S_1, S_2, \cdots S_n) = \tag{15}$$

$$\left\|\begin{array}{cccc} f(S_1^1, S_2^1, \cdots S_n^1) & & & \\ & f(S_1^2, S_2^2, \cdots S_n^2) & & \\ & & \begin{array}{c}\cdots\\ \cdots\\ \cdots\end{array} & \\ & & & f(S_1^p, S_2^p, \cdots S_n^p) \end{array}\right\|.$$

As an illustration, the matrix:

$$\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$$

has no matrices conjugate to it in the ordinary sense, but has as its generalized conjugates:

$$\begin{Vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

It is to be noticed that the method of constructing the generalized conjugate matrices leads to a unique result only in exceptional cases, like that just given. For example, in addition to the ordinary conjugates, the matrix:

$$\begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}$$

has as generalized conjugates any of the pairs:

$$\begin{Vmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{Vmatrix}, \quad \begin{Vmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{Vmatrix};$$

$$\begin{Vmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{Vmatrix}, \quad \begin{Vmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{Vmatrix};$$

or:

$$\begin{Vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{Vmatrix}, \quad \begin{Vmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{Vmatrix}.$$

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